Appendix B for "Political Selection and Persistence of Bad Governments": Omitted Proofs (Not for Publication)

Proof of Theorem 2. Part 1. We prove the statement for the case $l = 0$. The case $l \geq 1$ is covered by Part 2 of the theorem.

To obtain a contradiction, suppose that there is a cycle; this implies that so that there are $q \geq 2$ different governments $H_1, \ldots, H_q$ such that $\phi(H_j) = H_{j+1}$ for all $1 \leq j < q$, and $\phi(H_q) = H_1$. Without loss of generality, let H_1 be the least competent of these governments. Take $H = \phi(H_2)$ (if $q > 2$ then $H = H_3$ and if $q = 2$ then $H = H_1$). As ϕ is a political equilibrium, $V_i(H) > V_i(H_2)$ holds for a winning coalition in H_2. But winning coalitions are the same for all governments, so $V_i(H) > V_i(H_2)$ holds for a winning coalition in H_1. Moreover, by Assumption 1, $V_i(H) > w_i(H_1) / (1 - \beta)$ for all players except, perhaps, members of H_1 (as H_1 is the least competent government). However, the existence of such alternative H contradicts that ϕ is a political equilibrium, as the condition (ii) of Definition 1 is violated for government H_1. This contradiction completes the proof.

Part 2. Suppose to obtain a contradiction that there is a cycle, so that there are $q \geq 2$ different governments $H_1, \ldots, H_q$ such that $\phi(H_j) = H_{j+1}$ for all $1 \leq j < q$, and $\phi(H_q) = H_1$. Without loss of generality, let H_1 be the most competent of these governments. Take any $i \in H_1$. In that case, $V_i(H_1) > V_i(H_2)$, as player i gets the highest utility under H_1 (formally, we have $V_i(H_2) < w_i(H_1) / (1 - \beta)$ as H_1 is i's most preferred government in the cycle, hence $w_i(H_1) + \beta V_i(H_2) > V_i(H_2)$, which means $V_i(H_1) > V_i(H_2)$). Since this holds for all members of H_1 and $l_{H_1} \geq 1$, it is impossible that for a winning coalition of players in H_1 the condition $V_i(H_2) > V_i(H_1)$ is satisfied. This contradiction completes the proof.

Part 3. Suppose to obtain a contradiction that the statement does not hold. As the number of mappings $\mathcal{G} \to \mathcal{G}$ is finite, there exists mapping ϕ which forms a cyclic political equilibrium for β arbitrarily close to 1. Moreover, since the number of coalitions is finite, we can only consider β in which ϕ is supported by the same coalitions in players. Let $H_1, \ldots, H_q$ with $q \geq 2$ satisfy $\phi(H_j) = H_{j+1}$ for all $1 \leq j < q$ and $\phi(H_q) = H_1$, so that the sequence constitutes a cycle for ϕ. This means, in particular, that for any $G \in \{H_1, \ldots, H_q\}$, the inequality $V_i(\phi(G) \mid \phi, \beta) > V_i(G \mid \phi, \beta)$ is satisfied for the same winning coalition of players for some β arbitrarily close to 1. As this inequality is equivalent to $w_i(G) < (1 - \beta) V_i(\phi(G) \mid \phi, \beta)$, we must have (by taking the limit as $\beta \to 1$) that

$$w_i(G) \leq u_i \equiv \frac{1}{q} \sum_{p=1}^{q} w_i(H_p),$$

where the last the quality defines u_i.

Without loss of generality, suppose that H_2 is the least competent of the governments in the cycle. Consider first the case $q = 2$. We must have that

$$w_i(H_1) \leq \frac{w_i(H_1) + w_i(H_2)}{2},$$

and hence $w_i(H_1) \leq w_i(H_2)$, for a winning coalition of players in H_1. However, only members of H_2 may satisfy these inequalities, and their number is less than m_{H_1}. We get an immediate contradiction.

Consider now the more complicated case $q \geq 3$. In this case, H_1, H_2, H_3 are three different governments. We take $H = H_3$ and show that the condition (ii) of Definition 1 is violated for current government H_1 and alternative $H = H_3$. In particular, we need to check that the following two conditions are satisfied:

(1) $V_i(H_3) > V_i(H_2)$ for a winning coalition of players. Note that this is equivalent to $w_i(H_2) < V_i(H_3)$. The latter is satisfied for sufficiently large β provided that $w_i(H_2) < u_i$. This last inequality holds for all players except, perhaps, members of H_2, i.e., for at least m_{H_1} of them. Therefore, we just have to prove that $w_i(H_2) < u_i$ for at least l_{H_1} members of H_1. However, we know that $w_i(H_1) \leq u_i$ for at least l_{H_1} members of H_1. Moreover, since they belong to H_1 and $\Gamma_{H_2} < \Gamma_{H_1}$, we must have that $w_i(H_2) < w_i(H_1)$ by Assumption 1 for each of these players. But this immediately implies $w_i(H_2) < u_i$.

(2) $V_i(H_3) > w_i(H_1) / (1 - \beta)$ for a winning coalition of players. Suppose not; then there must exist player i such that $(1 - \beta) V_i(H_3) \leq w_i(H_1) < (1 - \beta) V_i(H_2)$ (since the latter inequality holds for a winning coalition of players). Taking the limit, we get $u_i \leq w_i(H_1) \leq u_i$, and hence $w_i(H_1) = u_i$. However, this contradicts the assumption.

Consequently, we have described $H = H_3$ for which the condition (ii) of Definition 1 is violated. This contradiction completes the proof.

Part 4. The proof of this part follows the proof of Part 3, except the steps involved in checking the condition $V_i(H_3) > w_i(H_1) / (1 - \beta)$ for a winning coalition of players (indeed, it is only in this step that the assumption in Part 3 was used). To check that last condition, again suppose that it did not hold. Then there must exist player i such that $(1 - \beta) V_i(H_3) \leq w_i(H_1) < (1 - \beta) V_i(H_2)$ (since the latter inequality holds for a winning coalition of players). Taking the limit, we get $u_i \leq w_i(H_1) \leq u_i$, and hence $w_i(H_1) = u_i$. Given the assumption, this is only possible if player i is either a member of all governments $H_1, \ldots, H_q$ or a member of none of them (otherwise $w_i(H_1) > u_i$ if $i \in H_1$ and $w_i(H_1) < u_i$ if $i \notin H_1$). In both cases, $w_i(H_2) < w_i(H_1)$, and thus $w_i(H_2) < u_i$. But then $V_i(H_3) > V_i(H_2)$ if β is sufficiently close to 1, establishing a contradiction. This contradiction completes the proof. ■

Proof of Proposition 1. Part 1. Suppose, to obtain a contradiction, that $|G \cap H| \geq l$, but $G \neq H$. By Assumption 3 we need to have $\gamma_G < \gamma_H$ or $\gamma_G > \gamma_H$; without loss of generality, assume the former. Then $H \succ G$ by Lemma 1, since $|G \cap H| \geq l$. Note that $G = G_q$ for some q and $H = G_j$ for some j such that $j < q$. Since H is stable, $\phi(G_j) = G_j$, but then $\mathcal{M}_q \neq \varnothing$ by (5), and so $\phi(G_q) \neq G_q$, as follows from (6). However, this contradicts the hypothesis that $G_q = G \in \mathcal{D}$, and thus completes the proof.

Part 2. By definition of mapping ϕ, $\phi(G_1) = G_1$, so $G_1 \in \mathcal{D}$. Take any government $G \in \mathcal{D}$; since $|G \cap G_1| \geq 0 = l$, we have $G = G_1$ by part 1. Consequently, $\mathcal{D} = \{G_1\}$, so $\mathcal{D}$ is a singleton. Now, for any G, $\phi(G) \in \mathcal{D}$, and thus $\phi(G) = G_1$.

Part 3. As before, the most competent government, G_1, is stable, i.e. $G_1 \in \mathcal{D}$. Now consider the set of governments which intersect with G_1 by fewer than l members:

$$\mathcal{B} = \{G \in \mathcal{G} : |G \cap G_1| < l\}.$$

This set is non-empty, because $n > 2k$ implies that there exists a government which does not intersect with G_1; obviously, it is in $\mathcal{B}$. Now take the most competent government from $\mathcal{B}$, G_j where

$$j = \min\{q : 1 \leq q \leq |G| \text{ and } G_q \in \mathcal{B}\}.$$

We have $G_j \neq G_1$, because $G_1 \notin \mathcal{B}$. Let us show that G_j is stable. Note that any government G_q such that $\gamma_{G_q} > \gamma_{G_j}$ does not belong to $\mathcal{B}$ and therefore has at least l common members with stable government G_1. Hence, $\phi(G_q) = G_1$ (see (6)), and therefore G_q is unstable, except for the case $q = 1$. Now we observe that set $\mathcal{M}_j$ is empty: for each government G_q with $1 < q \leq j$ either the first condition in (8) is violated (if $q = 1$) or the second one (otherwise). But this implies that $\phi(G_j) = G_j$, so G_j is stable. This proves that if $l \geq 1$, $\mathcal{D}$ contains at least two elements. Finally, note that this boundary is achieved: for example, if $l = 1$ and $n < 3k$.

Part 4. If $l = k$, then for any $G_q \in \mathcal{G}$, it is impossible that $H \succ G_q$ for some alternative $H \neq G_q$, as there will exist player $i \in G_q \setminus H$ for whom $w_i(H) < w_i(G_q)$. Hence, $\mathcal{M}_q = \varnothing$ for all q, and thus $\phi(G_q) = G_q$ by (6). Consequently, $\mathcal{D} = \mathcal{G}$, and this completes the proof of Proposition 1. ■

Proof of Proposition 2. Part 1. We prove the more general part 2, then the statement of part 1 will be a corollary: to obtain (11), one only needs to substitute $x = k$ into (12).

Part 2. Let us prove the existence of such stable government. Define a set-valued function $\chi : \mathcal{C}^l \to \mathcal{C}^{k-l} \cup \{\varnothing\}$ by

$$\chi(S) = \begin{cases} G \setminus S & \text{if } G \in \mathcal{D} \text{ and } S \subset G; \\ \varnothing & \text{if there exists no } G \in \mathcal{D} \text{ such that } S \subset G. \end{cases} \tag{B1}$$

In words, for any coalition of l individuals, function χ assigns a coalition of $k - l$ individuals such that their union is a stable government whenever such other coalition exists or an empty set when it does not exist. Note that $\chi(S)$ is a well defined single valued function: indeed, there cannot be two different stable governments G and H which contain S, for this would violate Proposition 1 (part 1), as they intersect by at least l members from S.

Let Y_{l-1} be some coalition of $l-1$ individuals such that $X \cap Y_{l-1} = \varnothing$; denote these individuals by $i_1, \ldots, i_{l-1}$. We will now add $k-l+1$ individuals $i_l, \ldots, i_k$ to this coalition one by one and we will denote the intermediate coalitions by $Y_l, \ldots, Y_k$, and then prove that Y_k satisfies the requirements. Let $X_{l-1} = X$, and let

$$X_l = (X \cup Y_{l-1}) \cup \left(\bigcup_{i \in X} \chi \left(Y_{l-1} \cup \{i\} \right) \right). \tag{B2}$$

Intuitively, we take the set of individuals which are either forbidden to join the government under construction by our requirements (X) or are already there (Y_{l-1}), and add all individuals which can be in the same government with all individuals from Y_{l-1} and at least one individual from $X_{l-1} = X$. Now take some individual $i_l \in \mathcal{I} \setminus X_l$ (below we show that such individual exists) and let $Y_l = Y_{l-1} \cup \{i_l\}$. At each subsequent step z, $l+1 \leq z \leq k$, we choose zth individual for the government under construction as follows. We first define

$$X_z = (X \cup Y_{z-1}) \cup \left(\bigcup_{S \subset Y_{z-1} : |S| = l-1; i \in X} \chi \left(S \cup \{i\} \right) \right) \tag{B3}$$

and then take

$$i_z \in \mathcal{I} \setminus X_z \tag{B4}$$

(we prove that we can do that later) and denote $Y_z = Y_{z-1} \cup \{i_z\}$. Let the last government obtained in this way be denoted by $Y = Y_k$.

We now show that $\phi(Y) \cap X = \varnothing$. Suppose not, then there is individual $i \in \phi(Y) \cap X$. By (6) we must have that $|\phi(Y) \cap Y| \geq l$; take the individual i_j with the highest j of such individuals. Clearly, $j \geq l$, so individual i_j could not be a member of the initial Y_{l-1} and was added at a later stage. Now let S be a subset of $(\phi(Y) \cap Y) \setminus \{i_j\}$ such that $|S| = l-1$. Since government $\phi(Y)$ is stable and contains the entire S as well as $i \in X$ (and $i \notin S$ because $S \subset Y$ and $X \cap Y = \varnothing$), we must have $\chi(S \cup \{i\}) = \phi(Y)$. Consequently, if we consider the right-hand side of (B3) for $z = j$, we will immediately get that $\phi(Y) \subset X_j$, and therefore $i_j \in X_j$. But we picked i_j such that $i_j \in \mathcal{I} \setminus X_j$, according to (B4). We get to a contradiction, which proves that $\phi(Y) \cap X = \varnothing$, so $\phi(Y)$ is a stable government which contains no member of X.

It remains to show that we can always pick such individual; we need to show that the number of individuals in X_z is less than n for any $z : l \leq z \leq k$. Note that the union in the inner parentheses of (B3) consists of at most

$$(k-l)\binom{z-1}{l-1}x \leq (k-l)\binom{k-1}{l-1}x$$

individuals, while $z-1 \leq k-1$. Therefore, it is sufficient to require that

$$\begin{aligned} n &> x+k-1+(k-l)\binom{k-1}{l-1}x \\ &= x+k-1+x(k-l)\frac{(k-1)!}{(l-1)!\,(k-l)!}. \end{aligned}$$

Because we are dealing with integers, this implies (12), which completes the proof.

Part 3. This follows immediately by using Assumption 4 and setting $\rho = x$ in (12), which gives (13). ■

Proof of Proposition 3. Part 1. By Assumption 2, $0 \leq l \leq k$, so either $l = 0$ or $l = 1$. If $l = 0$, then Proposition 1 (part 2) implies that the only stable government is G_1, so $\phi(G) = G_1$ for all $G \in \mathcal{G}$, where $G_1 = \{i_1\}$. If $l = 1$, then Proposition 1 (part 4) implies that any G is stable.

Part 2. In this case, either $l = 0$, $l = 1$, or $l = 2$. If $l = 0$ or $l = 2$, the proof is similar to that of part 1 and follows from Proposition 1 (parts 2 and 4). If $l = 1$, then $\{i_1, i_2\}$ is the most competent, and hence stable, government. By 1 (part 1), any other government containing i_1 or i_2 is unstable. Hence, $\{i_3, i_4\}$, the most competent government not containing i_1 or i_2, is stable. Proceeding likewise, we find that the only stable governments are $\{i_{2j-1}, i_{2j}\}$ for $1 \leq j \leq n/2$.

By the construction of mapping ϕ, either $\phi(G) = G$ or $|\phi(G) \cap G| = 1$. If $G = \{i_a, i_b\}$ with $a < b$, then $\phi(G)$ will include either i_a or i_b. Now it is evident that $\phi(G)$ will be the stable state which includes i_a, because it is more competent than the one which includes i_b if the latter exists and is different. ■

Proof of Proposition 4. The probability of having the most able player in the government under the royalty system is 1. Indeed, government $\phi(G)$ for any G consists of l irreplaceable members and $k-l$ most competent members. Since $l < k$, this always includes the most competent player. In the case of a junta-like system, there is a positive probability that a government that does not include player i_1 is stable. If $\frac{\gamma_1-\gamma_2}{\gamma_2-\gamma_n}$ is sufficiently large, any government that includes player i_1 is more competent that a government that does not. The first part follows. Now consider the probability that the least competent player, i_n, is a part of the government. In a royalty system, this will happen if and only if one of the initial government members (who is irreplaceable) is the least competent. In a junta-like system, if $\frac{\gamma_1-\gamma_{n-1}}{\gamma_{n-1}-\gamma_n}$ is low enough, a government which does not the least competent player i_n will never transit to a government that includes one. At the same time, it is possible that a government that includes the least competent player will remove him (take, for example, the best government with one of the players replaced by i_n). This means that the probability of having the least competent player in the government is higher under royalty than under junta. This completes the proof. ■

Proof of Theorem 3. The proof follows immediately from Theorem 1 and the assumption that changes are sufficiently infrequent. Indeed, in the latter case, all the strict inequalities in Definitions 1 and 2 are preserved. ■

Proof of Proposition 5. Part 1. If $l = 0$, then by Proposition 1 for any G, $\phi^t(G) = G_1^t$, where G_1^t is the most competent government $\{i_1^t, \ldots, i_k^t\}$.

Part 2. Suppose $l = 1$, then Proposition 1 provides a full characterization. There are $\lfloor n/k \rfloor \leq n/k$ stable governments. Each consists of k individuals, so the probability that a random new government coincides with any given stable government is $1/\binom{n}{k} = \frac{k!(n-k)!}{n!}$. The probability that it coincides with any stable government is $\lfloor n/k \rfloor / \binom{n}{k} \leq \frac{n}{k}\frac{k!(n-k)!}{n!} = \frac{(k-1)!(n-k)!}{(n-1)!} = 1/\binom{n-1}{k-1}$. The government will change to a more competent one if and only if it is unstable, which happens with probability greater than or equal to $1 - 1/\binom{n-1}{k-1}$.

The most competent government will be installed if and only if after the shock, the government contains at least 1 of the k most competent members. The probability that it does not contain any of these equals $\binom{n-k}{k}/\binom{n}{k}$ (this is the number of combinations that do not include k most competent members divided by the total number of combinations). We have

$$\begin{aligned}\binom{n-k}{k}/\binom{n}{k} &= \frac{(n-k)!k!\,(n-k!)}{k!\,(n-2k)!n!} \\ &= \frac{(n-k)!}{(n-2k)!}\frac{(n-k!)}{n!} \\ &= \prod\nolimits_{j=1}^{k} \frac{n-k-j}{n-j}.\end{aligned}$$

Since each of the k factors tends to 1 as $n \to \infty$, so does the product. Hence, the probability that the most competent government will arise, $\pi_t(l, k, n \mid G, \{\Gamma_G\}) = 1 - \binom{n-k}{k}/\binom{n}{k}$, tends to 0 as $n \to \infty$.

Part 3. If $l = k$, then $\phi^t(G) = G$ for any t and G. Hence, the government will not change. It will be the most competent if it contains k most competent individuals, which happens with probability $1/\binom{n}{k}$. This is less than 1, which is the corresponding probability for $l = 0$, so $\pi_t(l = k, k, n \mid G, \{\Gamma_G\}) < \pi_t(l = 0, k, n \mid G, \{\Gamma_G\})$. If $k \geq 2$, it is also less than the corresponding probability for $l = 1$: in the latter case, there are at least two governments which will lead to the most competent one: $\{i_1^t, \ldots, i_k^t\}$ and $\{i_1^t, \ldots, i_{k-1}^t, i_{k+1}^t\}$, i.e., $\pi_t(l = 0, k, n \mid G, \{\Gamma_G\}) \geq 2/\binom{n}{k} > \pi_t(l = 0, k, n \mid G, \{\Gamma_G\})$. This completes the proof. ■

Proof of Proposition 6. Given the specific changes in $\{\Gamma_G\}$ in this case, the probability of having the most competent government G_1^t (for any initial G and $\{\Gamma_G\}$) is the probability that at least l members of G^t are members of G_1^t. This probability equals (from hypergeometric distribution):

$$\pi_t(l,k,n \mid G,\{\Gamma_G\}) = \frac{\sum_{q=l}^{k}\binom{k}{q}\binom{n-k}{k-q}}{\binom{n}{k}},$$

and is strictly decreasing in l. ■

Proof of Proposition 7. Part 1. Any such swapping (or, more generally, any transposition σ, where $\sigma(i)$ is the individual whose former competence individual i now has) induces a one-to-one mapping that maps government G to government $\rho(G)$: $i \in \rho(G)$ if and only if $\sigma(i) \in G$. By construction, $\Gamma_G^{t-1} = \Gamma_{\rho(G)}^t$, and, by construction of mapping ϕ, $\phi^{t-1}(G) = \phi^t(\rho(G))$ for all G. If all transitions occur in one stage, and a shock triggers a period of instability, then with probability 1 all shocks arrive at times t where government G^{t-1} is ϕ^{t-1}-stable.

If abilities of only two individuals are swapped, then $|G \cap \rho(G)| \geq k-1 \geq l$. But G is ϕ^{t-1}-stable with probability 1, hence, $\rho(G)$ is ϕ^t-stable. Consider two cases. If $\Gamma_G^t \geq \Gamma_{\rho(G)}^t$, then $\Gamma_{\phi^t(G)}^t \geq \Gamma_G^t \geq \Gamma_{\rho(G)}^t$. If $\Gamma_G^t < \Gamma_{\rho(G)}^t$, then again $\Gamma_{\phi^t(G)}^t \geq \Gamma_{\rho(G)}^t$, since there is a ϕ^t-stable government $\rho(G)$ which has with G at least l common members and the competence of which is $\Gamma_{\rho(G)}^t$. Hence, $\phi^t(G)$ is either $\rho(G)$ or a more competent government. Hence, the competence of government cannot decrease. However, it may increase, unless G contains k most competent members. Indeed, in that case there exist $i,j \in \mathcal{I}$ with $i<j$ such that $i \notin G$ and $j \in G$. Obviously, swapping the abilities of these individuals increases the competence of G: $\Gamma_G^t > \Gamma_G^{t-1}$, and thus the stable government that will evolve will satisfy $\Gamma_{\phi(G)}^t \geq \Gamma_G^t > \Gamma_G^{t-1}$. Since the probability of this swapping is non-zero, eventually the competence of government will improve. Since there is a finite number of possible values of current government's competence, then with probability 1 the most competent government will emerge.

Part 2. This follows from an argument of part 1, taking into account that if abilities of x individuals changed, then $|G \cap \rho(G)| \geq k - \lfloor x/2 \rfloor \geq l$. Indeed, if $|G \cap \rho(G)| < k - \lfloor x/2 \rfloor$, we would have $|G \cap \rho(G)| \leq k - \lfloor (x+1)/2 \rfloor$ since the numbers of both sides are integers, and thus $|(G \setminus \rho(G)) \cup (\rho(G) \setminus G)| > 2\lfloor (x+1)/2 \rfloor \geq x$. However, all individuals in $(G \setminus \rho(G)) \cup (\rho(G) \setminus G)$ changed their abilities, so the last inequality contradicts the assumption that no more than x individuals did. This contradiction completes the proof. ■

Proof of Theorem 5. Part 1. In part 1 of Theorem 4, we proved that for any $\xi \in \mathcal{X}$ there exists a MPE in pure strategies, and from part 2 of Theorem 4 it follows that these MPE constructed for different $\xi \in \mathcal{X}$ have the same equilibrium path of governments. The existence of an order-independent equilibrium follows.

Part 2. Suppose, to obtain a contradiction, that order-independent MPE in pure strategies σ^* is cyclic. Define mapping $\chi : \mathcal{G} \to \mathcal{G}$ as follows: $\chi(G) = H$ if for any node on equilibrium path which starts with government $G^t = G$ and $\nu^t = u$, the next government $G^{t+1} = H$. Since the equilibrium is in pure strategies, this mapping is well defined and unique. The assumption that equilibrium σ^* is acyclic implies that there is a sequence of pair-wise different governments $H_1, H_2, \ldots, H_q$ (where $q \geq 2$) such that $\chi(H_j) = H_{j+1}$ for $1 \leq j < q$ and $\chi(H_q) = H_1$. Without loss of generality, assume that H_2 has the least competence of all governments $H_1, H_2, \ldots, H_q$. If $q = 2$, then the cycle has two elements, of which H_2 is the worse government. However, this implies that H_2 cannot defeat H_1 even if it wins the primaries, since all players, except, perhaps, those in $H_2 \setminus H_1$, prefer H_1 to an eternal cycle of H_1 and H_2. This immediate contradiction implies that we only need to consider the case $q \geq 3$.

If $q \geq 3$, then, by the choice of H_2, $\Gamma_{H_1} > \Gamma_{H_2}$ and $\Gamma_{H_3} > \Gamma_{H_2}$. Without loss of generality, we may assume that the protocol is such that if the incumbent government is H_1, H_3 is put at the end (if H_3 is nominated); this is possible since σ^* is an order-independent equilibrium. By definition, we must have that proposal H_2 is nominated and accepted in this equilibrium along the equilibrium path.

Let us first prove that alternative H_3 will defeat the incumbent government H_1 if it wins the primaries. Consider a player i who would have weakly preferred H_2, the next equilibrium government, to win over H_1 if H_2 won the primaries; since H_2 defeats H_1 on the equilibrium path, such players must form a winning coalition in H_1. If $i \notin H_2$, then H_2 brings i the lowest utility of all governments in the cycle; hence, i would be willing to skip H_2; hence, such i would be strictly better off if H_3 defeated H_1. Now suppose $i \in H_2$. If, in addition, $i \in H_1$, then he prefers H_1 to H_2. Assume, to obtain a contradiction, that i weakly prefers that H_3 does not defeat H_1; it is then easy to see that since he prefers H_1 to H_2, he would strictly prefer H_2 not to defeat H_1 if H_2 won the primaries. The last case to consider is $i \in H_2$ and $i \notin H_1$. If β is sufficiently close to 1, then, as implied by Assumption $3'$, player i will either prefer that both H_2 and H_3 defeat H_1 or that none of them does. Consequently, all players who would support H_2 also support H_3, which proves that H_3 would be accepted if nominated.

Suppose to obtain a contradiction that H_3 is nominated in equilibrium. Then H_2 cannot win the primaries: in the last voting, H_2 must face H_3, and since, as we showed, only members of H_2 may prefer that H_2 rather than H_3 is the next government, H_3 must defeat H_2 in this voting. This means that in equilibrium H_3 is not nominated.

Suppose next that if all alternatives were nominated, some government G wins the primaries. It must then be the case that G defeats H_1: indeed, if instead H_1 would stay in power, then $G \neq H_3$ (we know that H_3 would defeat H_1), and this implies that in the last voting of the primaries, H_3 would defeat G. Let us denote the continuation utility that player i gets if some government H comes to power as $v_i(H)$. If there is at least one player with $v_i(G) > v_i(H_2)$, then this player has a profitable deviation during nominations: he can nominate all alternatives and ensure that G wins the primaries and defeats H_1. Otherwise, if $v_i(G) \leq v_i(H_2)$ for all players, we must have that $v_i(G) < v_i(H_3)$ for a winning coalition of players, which again means that G cannot win the primaries. This contradiction proves that for the protocol we chose, H_2 cannot be the next government, and this implies that there are no cyclic order-independent equilibria in pure strategies.

Part 3. The proof is similar to the proof of part 2. We define mapping χ in the same way and choose government H such that $\chi(\chi(H)) \neq \chi(H)$, but $\chi(\chi(\chi(H))) = \chi(\chi(H))$. We then take a protocol which puts government $\chi(\chi(H))$ at the end whenever it is nominated and come to a similar contradiction.

Part 4. This follows from part 3, since the only transition may happen at $t = 0$. ■

Proof of Theorem 6. Part 1. If δ is sufficiently small, then the ordering of continuation utilities for each player at the end of any period is the same as before, and the equilibrium constructed in the proof of part 1 of Theorem 4 proves this statement as well.

Part 2. If δ is sufficiently small, the proof of Theorem 5 (parts 2 and 3) may be applied here with minimal changes, which are omitted. ■